The Chow Ring of the Moduli Space of $g=0$ Prestable Curves.

(joint work in progress with Johannes Schmitt)

Younghan Bae

23.09.2020

§1. Introduction

k = base field

Let $\mathfrak{M}_{g,n}$: moduli space of prestable curves of genus g, n markings. Objects over S is

$$\mathfrak{M}_{g,n}(S) = \left\{ \begin{array}{c} C \\ \pi \downarrow \uparrow p_1, \ldots, p_n \\ S \end{array} \;\middle|\; \begin{array}{l} \pi : \text{flat, proper, representable} \\ \text{fibers are connected, reduced} \\ \text{curve w/ nodal singularity} \end{array} \right\}$$

$\mathfrak{M}_{g,n}$: algebraic stack, quasi-separated, smooth, locally of finite type / k.
(no stability condition)

- when $2g-2+n>0$,

$$\overline{\mathcal{M}}_{g,n} \hookrightarrow \mathfrak{M}_{g,n} \quad \text{open substack.}$$

⚠ We don't have $\int_{\mathfrak{M}_{g,n}} (\,\cdot\, -)$.

- This space has a well-defined cycle theory

$$CH_*(\mathcal{M}_{g,n})$$

by Kresch.

- We are interested in the tautological subring

$$R^*(\mathcal{M}_{g,n}) \subset CH^*(\mathcal{M}_{g,n})_{\mathbb{Q}}.$$

- In this talk, we are mainly interested in $g=0$ case.

<u>Thm A</u>. When $g=0$, $R^*(\mathcal{M}_{0,n}) = CH^*(\mathcal{M}_{0,n})$

↑ presented in this seminar last year.

<u>Thm B</u> When $g=0$, taut. relations are additively generated by the WDVV-relation & ψ, κ-relations.

↑ we will make this in a precise form.

§2. Cycle theory of algebraic stacks

(2.1) A. Kresch's cycle theory

- X = finite type scheme, DM-stack. Then a class in $CH_*(X)_{\mathbb{Q}}$ is represented by integral closed substack. [Vistoli]

- $X = [Y/G]$. Then take a finite approximation of $Y \times_G EG$. and

$$CH_*(X) := CH_*(\text{finite app. of } Y \times_G EG).$$

[Edidin - Graham, 98]

<u>Non-example</u> Let $\mathcal{M}_{0,0}^{\leq 2} \subset \mathcal{M}_{0,0}$ be the locus where curves have at most two nodes.

<u>Thm</u> [Kresch, 13] $\mathcal{M}_{0,0}^{\leq 2}$ is <u>not</u> a quotient stack.

Idea of Kresch :

$CH^{\circ}_{*}(-)$: cycles generated by integral closed substack.

$\widehat{CH}_{*}(-) := \varinjlim_{E} CH^{\circ}_{*+rkE}(E)$ where E: vector bundle on a space

Def (Kresch)

$$CH_*(X) := \varinjlim_{\substack{Y \to X \\ \text{projective}}} \widehat{CH}_*(Y) / \hat{B}(Y)$$

(f, α) $f: Y \to X$, $E \to Y$. $\alpha \in CH^{\circ}(E)$

- If X is a finite type $/k$, stratified by locally closed substacks which are quotient stack$^{(*)}$. Then $CH_*(-)$ has projective pushforward, flat pullback, Chern classes & refined Gysin pullback along lci morphisms

(*) local condition : stabilizer group of each geom. pt is affine

* We exclude $(g,n) = (1,0)$

If X : locally finite type / k, take a directed system of open covers $\{U_i\}$. U_i : finite type / k.

$$CH_*(X) := \varprojlim CH_*(U_i)$$

(2.2) Proper pushforward (after Skowera).

It took a while to define pushforward cycles along proper (but not nec. proj) morphisms.

$f: X \to Y$ projective if

$$\begin{tikzcd} X \arrow[r, hook, "\text{closed}"] \arrow[d, "f"'] & \mathbb{P}(\mathcal{E}) \arrow[dl] \\ Y & \end{tikzcd}$$

$\mathcal{E}$ coherent sheaf on Y

Example (Fulghesu) The projection

$$\pi: C \longrightarrow \mathcal{M}_{0,0}^{\leq 2}$$

from the universal curve is not projective.

Thm (Skowera '19) Let $f: X \to Y$ proper, representable. Then there exists

$$f_*: CH_*(X, \mathbb{Z}) \longrightarrow CH_*(Y, \mathbb{Z})$$

If f is proper, relatively DM-type,

$$f_*: CH_*(X, \mathbb{Q}) \longrightarrow CH_*(Y, \mathbb{Q}).$$

Moreover f_* is compatible with flat pullback, Gysin pullback etc.

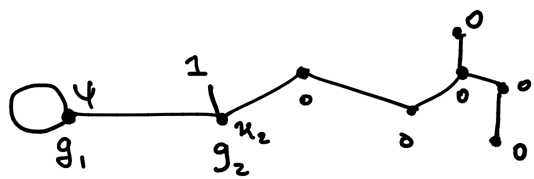
§3. Proof of Theorem A.

(3.1) Tautological classes

- We showed that a parallel construction of tautological rings for $\mathcal{M}_{g,n}$ works.
  - ↳ Moduli presentation of $\mathcal{C}_{g,n} \to \mathcal{M}_{g,n}$.
- Additive basis of $R^*(\mathcal{M}_{g,n})$:

$[\Gamma, \alpha]$ Γ : prestable graph of genus g, n legs

α : ψ & κ - classes.

<u>Ex</u> $[\Gamma, \alpha] =$ 

(3.2) Proof of Thm A

Thm A $R^d(\mathcal{M}_{0,n}) = CH^d(\mathcal{M}_{0,n})_{\mathbb{Q}}$ $\forall n \geq 0$ $\forall d$.

$\hookrightarrow$ for simplicity, $n \geq 1$.

- We use the recursive boundary structure of $\mathcal{M}_{0,n}$ and induction on d.

- We start from $\mathcal{M}_{0,n}^{sm}$ $\leftarrow$ locus where C is smooth.

(a) $\mathcal{M}_{0,1}^{sm} = BU$, $U = \mathbb{G}_a \rtimes \mathbb{G}_m$: group of affine transformations of $\mathbb{A}^1$

$\rightsquigarrow CH^*(\mathcal{M}_{0,1}^{sm}) = \mathbb{Q}[\psi_1]$

(b) $\mathcal{M}_{0,2}^{sm} = B\mathbb{G}_m$.

$\rightsquigarrow CH^*(\mathcal{M}_{0,2}^{sm}) = \mathbb{Q}[\psi_1]$

(c) $\mathcal{M}_{0,n}^{sm} = M_{0,n}$, $n \geq 3$. $\rightsquigarrow CH^*(M_{0,n}) = \mathbb{Q}\langle M_{0,n}\rangle$

Pf) (b) $\Rightarrow$ (a). Use the homotopy invariance for affine bundles. $\boxed{}$

In the proof we use two ingredients ;

(i) $\mathcal{M}_{0,n}^{sm}$ satisfies the Chow Künneth property

(ii) If $f : X \to Y$ proper, surjective, relative DM.

then

$$f_* : CH_*(X)_{\mathbb{Q}} \to CH_*(Y)_{\mathbb{Q}} \quad \text{surjective}$$

Sketch of the proof of Thm A)

Consider the excision sequence

$$\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim}{CH^{d-1}(\partial\mathcal{M}_{0,n})} \to CH^{d}(\mathcal{M}_{0,n}) \to \underbrace{CH^d(\mathcal{M}_{0,n}^{sm})}_{\text{tautological.}} \to 0$$

We have the gluing map.

$$\coprod_{I \subset [n]} \mathcal{M}_{0,I\cup\{p\}} \times \mathcal{M}_{0,I^c\cup\{p'\}} \longrightarrow \partial\mathcal{M}_{0,n}$$

which is proper, representable, surjective.

$\Rightarrow$ pushforward is surjective.

- Easier to consider the Chow-Künneth generating property (CKgP). Namely, X satisfies CKgP if for any Y

$$CH_*(X) \otimes CH_*(Y) \longrightarrow CH_*(X \times Y)$$

is surjective.

- In particular (i), (ii) + excision seq

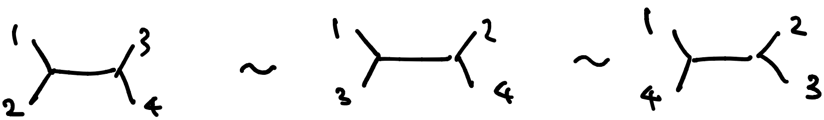
$\Rightarrow \mathcal{M}_{0,n}$ has CKgP.

- $\underbrace{CH_*(\mathcal{M}_{0,n_1}) \otimes CH_*(\mathcal{M}_{0,n_2})}_{\text{tautological by induction hypothesis}} \twoheadrightarrow CH_*(\mathcal{M}_{0,n_1} \times \mathcal{M}_{0,n_2})$

$\boxtimes$

§4. Revisit the WDVV-relation.

<u>Recall</u> (WDVV) $\overline{M}_{0,4} \simeq \mathbb{P}^1$. So



in $CH^1(\overline{M}_{0,4})$.

We will follow a nontraditional way to understand WDVV.

(4.1) Localization sequence.

Consider the localization sequence

$$CH_*(M_{0,4},1) \xrightarrow{\partial} CH_*(\partial\overline{M}_{0,4}) \longrightarrow CH_*(\overline{M}_{0,4}) \to CH_*(M_{0,4}) \to 0$$

where $CH_*(-,1)$ is the 1^{st} higher Chow group.

· Understand WDVV as the $\operatorname{Im}\partial$.

For 1^{st} higher Chow groups, we can forget about the transversality issue. Let U : scheme / k.

Let Δ^1 : algebraic 1-simplex. $R = \Delta^1 - \{0,1\}$.
$(\simeq \mathbb{A}^1_k)$

$$Z^*(U \times \Delta^2)^{prop} \xrightarrow{\partial} \underset{\substack{\cup \\ [W]}}{Z^*(U \times R)} \xrightarrow{\partial} Z^*(U) \to 0$$

$$\partial[W] := \underbrace{\overline{W}}_{\text{closure in } U \times \Delta^1} \cap (U \times [0] - U \times [1])$$

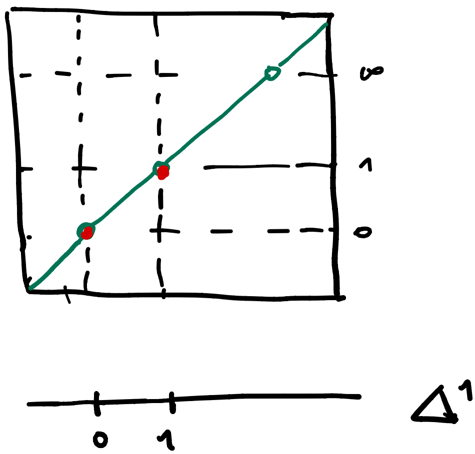
$Z^*(\cdot\cdot)^{prop} \subset Z^*(U \times \Delta^2)$ where the cycles intersect faces of $U \times \Delta^2$ in the right dimension.

$$CH^*(U,1) = \frac{\ker(\partial : Z^*(U \times R) \to Z^*(U))}{\operatorname{Im}(\partial : Z^*(U \times \Delta^2)^{prop} \to Z^*(U \times R))}$$

Proof of WDVV when $n=4$)

Let $[L_0] \in CH^1(\mathcal{M}_{0,4,1})$ corresponds to

$L_0 = \overline{(0,0),(1,1)}$



$\mathbb{P}^1 - \{0,1,\infty\} \cong \mathcal{M}_{0,4}$

Now one can compute

$$\partial[L_0] = [0] - [1] \quad \text{in } CH_0(\partial\overline{M}_{0,4})$$

$\boxtimes$

(4.2) General case.

For $n \geq 4$, we the corresponding motive

$$M_{gm}(M_{0,n}) \in DM^{eff}_{gm}(k)$$

and its motivic cohomology. (equivalently, its higher Chow groups)

- If U : sm scheme $/k$, then

$$CH^1(U,1) \cong H^1(U, \mathbb{Z}(1)) \cong H^0(U, \mathcal{O}_U^*) .$$

<u>Thm</u> [Chatzistamatiou, '07].

Let $U \subseteq \mathbb{A}^N_k$ be a hyperplane complement.

Then the motivic cohomology of U is generated by $H^1(U, \mathbb{Z}(1))$ over $H^*(k, \mathbb{Z}(\bullet))$. In particular

$$CH^l(U,1) = \begin{cases} H^0(U, \mathcal{O}_U^*) & \text{if } l = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Prop For $n \geq 4$, the image of the coboundary

$$CH^{\ell+1}(\overset{\mathfrak{M}^{sm}_{0,n}}{\overset{\shortparallel}{\mathcal{M}_{0,n}}}, 1) \xrightarrow{\partial} CH^{\ell}(\partial \mathfrak{M}_{0,n}) \to CH^{\ell+1}(\mathfrak{M}_{0,n})$$

is the set of WDVV relations if $\ell = 0$ and trivial if $\ell > 0$.

Pf) For simplicity, we prove this for $\mathcal{M}_{0,n} \subset \overline{\mathcal{M}}_{0,n}$.

Idea We want to pullback the previous computation along the forgetful morphism π

$$\begin{tikzcd}
\mathcal{M}_{0,n} \arrow[dr, hook] \arrow[d, hook, "i"'] & & \partial\overline{\mathcal{M}}_{0,n} \arrow[dl, hook'] \\
\pi^{-1}\mathcal{M}_{0,4} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \overline{\mathcal{M}}_{0,n} \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \pi^{-1}(\partial\overline{\mathcal{M}}_{0,4}) \arrow[l, hook'] \arrow[d] \arrow[u, hook, "j'"'] \\
\mathcal{M}_{0,4} \arrow[r, hook] & \overline{\mathcal{M}}_{0,4} & \partial\overline{\mathcal{M}}_{0,4} \arrow[l, hook']
\end{tikzcd}$$

$\pi^{-1}(M_{0,4})$ contains $M_{0,n}$ as an open set.

Let $j' : \pi^{-1}(\partial \overline{M}_{0,4}) \hookrightarrow \partial \overline{M}_{0,n}$: closed embedding.

We check :

$$\begin{tikzcd}
CH^1(M_{0,n}, 1) \arrow[r, "\partial"', "\text{WDVV}"] & CH^0(\partial \overline{M}_{0,n}, 1) \\
CH^1(\pi^{-1} M_{0,4}, 1) \arrow[u, "\iota^*"] \arrow[r, "\partial"] \arrow[ur, phantom, "\circlearrowright"] & CH^0(\pi^{-1} \partial \overline{M}_{0,4}, 1) \arrow[u, "j'^*"'] \\
CH^1(M_{0,4}, 1) \arrow[u, "\pi^*"] \arrow[r, "\partial", "\text{WDVV}"'] & CH^0(\partial \overline{M}_{0,4}) \arrow[u, "\pi^*"']
\end{tikzcd}$$

Using $S_n \circlearrowright \overline{M}_{0,n}$, any generator of $CH^1(M_{0,n}, 1)$ is a pullback of a class in $CH^1(M_{0,4}, 1)$

$\blacksquare$

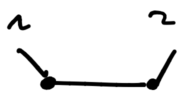
§5. Proof of Thm B.

Thm B Tautological relations are additively generated by the WDVV relation & $\psi.\kappa$-relations

↳ Restricting to $\overline{M}_{0,n}$, the result specializes to the work of Keel.

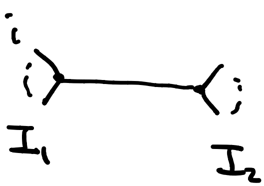
(5.1) $\psi.\kappa$ - relations.

We can simplify $\psi.\kappa$-monomials from the following relations

(a) $\psi_1 + \psi_2 =$ 

in $CH^1(\mathcal{M}_{0,2})$

(use the excision seq)

(b) $\psi_i = \sum\limits_{\substack{I_1 \sqcup I_2 = [n] \\ i \in I_1,\ j,k \in I_2}}$ 

in $CH^1(\mathcal{M}_{0,n})$, $n \geq 3$

(c) $\kappa_a = \sum \psi$ & boundary strata. $n \geq 1$.

(5.2) Strata space

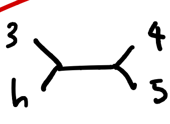
Let $S_{0,n}$ be the $g=0$ strata space.

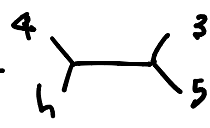
(i.e. formal linear sum of $[\Gamma, \alpha]$) $\leftarrow$ No multiplication yet.

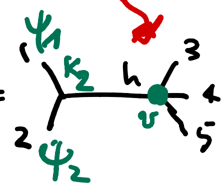
$\rightsquigarrow$ $S_{0,n} = \bigoplus_{p \geq 0} S^{p}_{0,n}$ $\quad$ $p = \#$ of edges.

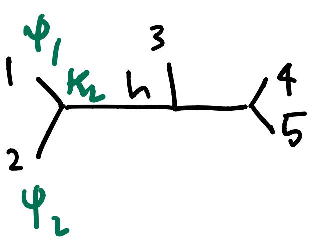
Def Let $R_0 \in S_{0,n_0}$. The set of relations in $S_{0,n}$ generated by R_0 is a $\mathbb{Q}$-subvector space of $S_{0,n}$ obtained by

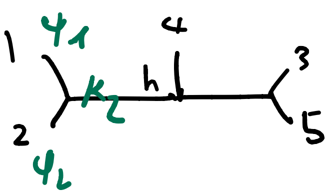
- Γ : prestable graph $\in S_{0,n}$
- $v \in V(\Gamma)$, identification of n_0 halfedges attached to v.
- gluing of R_0 at v

Ex $R_0 =$ 

$-$ 

, $\Gamma =$ 

$\rightsquigarrow$ glue 

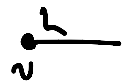
$-$ 

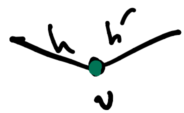
Let $R_{\kappa,\psi}$ be the set of relations of κ, ψ monomials obtained by (a) — (c).

<u>Def</u> Given a graph Γ, an element

$$\alpha = \prod_{v \in V(\Gamma)} \alpha_v$$

is said to be a <u>normal form</u> if:

(i) $n(v) = 1 \Rightarrow \alpha_v = \psi_h^b$ 

(ii) $n(v) = 2 \Rightarrow \alpha_v = \psi_h^c + (-\psi_{h'})^c$ 

(iii) $n(v) \geq 3 \Rightarrow \alpha_v = 1.$

"reduced"

- Let R_{WDVV} : relations obtained by gluing WDVV relations into normal form
- Let $S_{0,n}^{nf} \subset S_{0,n}$ be the subvector space additively generated by normal forms. Then

$$S_{0,n}^{nf} \hookrightarrow S_{0,n} \twoheadrightarrow S_{0,n}/R_{\kappa,\psi}$$

is surjective.

(5.3) Proof of Thm B.

- $S_{0,n} \longrightarrow CH^*(M_{0,n})$. $[\Gamma.\alpha] \longmapsto \xi_{\Gamma *}(\alpha)$.

Thm B' $CH^*(M_{0,n}) \cong S_{0,n} / (R_{K.Y} + R_{WDVV})$.

Simple diagram chasing reduces the question to show that the kernel of

$$S_{0,n}^{nf} \longrightarrow CH^*(M_{0,n})$$

is R_{WDVV}.

Step 1 We stratify $M_{0,n}$ as follows.

$$M_{0,n}^{\geq p} = \{ C \mid C \text{ has at least } p \text{ nodes} \} \overset{\text{closed}}{\hookrightarrow} M_{0,n}$$

$$M_{0,n}^{\geq p} \setminus M_{0,n}^{\geq p+1} = M_{0,n}^{=p} \leftarrow \text{exactly } p \text{ nodes}$$

$$\cong \coprod_{\Gamma \in G_p} \left(\prod_{v \in V(\Gamma)} M_{0,n(v)}^{sm} / \operatorname{Aut} \Gamma \right)$$

where G_p : set of prestable graphs with p edges.

The localization seq reads :

$$\boxed{CH^d(\mathcal{M}_{0,n}^{=p}, 1) \xrightarrow{\partial}} CH^{d-1}(\mathcal{M}_{0,n}^{\geq p+1}) \to CH^d(\mathcal{M}_{0,n}^{\geq p}) \to CH^d(\mathcal{M}_{0,n}^{=p}) \to 0$$

$$\mathcal{M}^{\geq p+1} \overset{\text{closed}}{\hookrightarrow} \mathcal{M}^{\geq p}$$

Step 2. $\mathcal{M}_{0,n}^{sm}$ satisfies the Chow-Künneth property for $CH^*(-,1)$: For Y : quotient stack,

$$\underbrace{CH_*(\mathcal{M}_{0,n}^{sm}, \bullet) \otimes CH_*(Y, \bullet)}_{CH_*(k,\bullet)} \longrightarrow CH_*(\mathcal{M}_{0,n}^{sm} \times Y, \bullet)$$

is an isomorphism in deg $\bullet = 1$.

In our setting, $CH^*(\mathcal{M}_{0,n}^{=p}, 1)$ is isomorphic to

$$\bigoplus_{\Gamma \in G_p} \left[\bigoplus_{\substack{v \in V(\Gamma) \\ n(v) \geq 3}} \left(\underset{CH^1(\mathcal{M}_{0,n(v)}, 1)}{\underset{\shortparallel}{CH^1(\mathcal{M}_{0,n(v)}^{sm}, 1)}} \otimes \bigotimes_{v' \neq v} CH^*(\mathcal{M}_{0,n(v')}^{sm}) \right) \right]^{\mathrm{Aut}\,\Gamma}.$$

$*$ If all $n(v) \leq 2$. $k^* \otimes \bigotimes_{v} CH^*(\mathcal{M}_{0,n(v')}^{sm})$

Moreover,

$$\partial\Big(\alpha_v \otimes \bigotimes_{v'\neq v} \alpha_{v'}\Big) = \partial(\alpha_v) \otimes \overline{\alpha}_{v'}$$

where $\overline{\alpha}_{v'}$ is any extension of $\alpha_{v'}$. (this formula is independent of $\overline{\alpha}_{v'}$)

We proved:

Prop. The image of

$$\partial : CH^*(\mathcal{M}_{0,n}^{=p}, 1) \longrightarrow CH^{*-1}(\mathcal{M}_{0,n}^{\geq p+1})$$

is the same as the image of

$$R^{p+1}_{WDVV} \longrightarrow S^{nf.\,p+1}_{0,n} \longrightarrow CH^*(\mathcal{M}_{0,n}^{\geq p+1}).$$

Step 3.

It is not so hard to prove that

$$S^{nf.p}_{0,n} \to CH^*(M^{\geq p}_{0,n}) \longrightarrow CH^*(M^{=p}_{0,n})$$

is an isomorphism. So we have a splitting :

$$\begin{tikzcd}
 & & & S^{nf.p}_{0,n} \arrow[dl] \arrow[d, "\cong"] & \\
CH^*(M^{=p}_{0,n}.1) \arrow[r, "\partial"] & CH^{*-1}(M^{\geq p+1}_{0,n}) \arrow[r] & CH^*(M^{\geq p}_{0,n}) \arrow[r] & CH^*(M^{=p}_{0,n}) \arrow[r] & 0
\end{tikzcd}$$

$$\Rightarrow \quad CH^*(M^{\geq p}_{0,n}) \cong S^{nf.p}_{0,n} \oplus CH^{*-1}(M^{\geq p+1}_{0,n}) / CH^*(M^{=p}_{0,n}.1)$$

$$\cong S^{nf.p}_{0,n} \oplus CH^{*-1}(M^{\geq p+1}_{0,n}) / R^{p+1}_{\mathrm{WDVV}} .$$

The localization sequence for $p = 0, 1, \dots$ yields

$$
\begin{aligned}
CH^*(\mathcal{M}_{0,n}) &\cong S^{nf,0}_{0,n} \oplus CH^{*-1}(\mathcal{M}^{\geq 1}_{0,n}) / R^1_{WDVV} \\
&\cong S^{nf,0}_{0,n} \oplus (S^{nf,1}_{0,n} / R^1_{WDVV}) \oplus CH^{*-2}(\mathcal{M}^{\geq 2}_{0,n}) / R^2_{WDVV} \\
&\ \ \vdots \\
&\cong S^{nf}_{0,n} / R_{WDVV}.
\end{aligned}
$$

This proves

$$CH^*(\mathcal{M}_{0,n}) \cong S_{0,n} / (R_{\psi,K} + R_{WDVV})$$

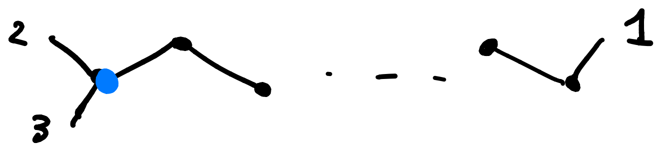
$\boxtimes$

§6. Remarks

(6.1) Previous results

- [Oesinghaus '18] $\mathcal{T} \subset \mathcal{M}_{0,3}$ open substack

Γ =



He constructed an atlas $[\mathbb{A}^n / \mathbb{G}_m^n] \to \mathcal{T}$ and showed

$$\Longrightarrow \quad CH_*(\mathcal{T}) \cong QSym_{\mathbb{Z}_{>0}}$$

We can identify basis of $QSym_{\mathbb{Z}_{>0}}$ with tautological classes.

Eg $J = (\hat{j}_1, \ldots, \hat{j}$

- [Fulghesu, '10] When $p \le 3$, he computed $CH^*(\mathcal{M}_{0,0}^{\le p})$ using explicit generators & relations

e.g $CH^*(\mathcal{M}_{0,0}^{\le 3}) \cong \mathbb{Q}$ - graded algebra with 10 generators and 11 relations

- It is not so easy to write his classes as tautological classes.

- It is possible to compare

$$\dim CH^d(\mathcal{M}_{0,0}^{\le p})$$

from [Fulghesu] & ours.

$p \le 2$: [Fulghesu] match with ours.

$p = 3$, $d \ge 8$: $\dim CH^8(\mathcal{M}_{0,0}^{\le 3}) = \begin{cases} \underline{55} \leftarrow \text{[Fulghesu]} \\ \underline{54} \leftarrow \text{ours} \end{cases}$

We think [Fulghesu] is missing at least one relation in $CH^8(\mathcal{M}_{0,0}^{\le 3})$.

(6.2) $\overline{M}_{g,n}$ vs $\mathfrak{M}_{g,n}$.

If $2g-2+n>0$, $\mathfrak{M}_{g,n}$ has a retraction map to $\overline{M}_{g,n}$.

$$\begin{tikzcd}
\overline{M}_{g,n} \arrow[r, hook, "i"] \arrow[rr, bend right, "id."'] & \mathfrak{M}_{g,n} \arrow[r, "st"] & \overline{M}_{g,n}
\end{tikzcd}$$

st: stabilization map (flat).

$\rightsquigarrow CH_*(\overline{M}_{g,n}) \hookrightarrow CH_*(\mathfrak{M}_{g,n})$.

Can we understand $R^*(\mathfrak{M}_{g,n})$ from $R^*(\overline{M}_{g,n})$ and $CH^*(\mathfrak{M}_{0,n})$?

When $* = 1$.

<u>Prop.</u> $CH^1(\mathfrak{M}_{g,n}) = R^1(\mathfrak{M}_{g,n})$ and all tautological relations are pullbacked from $\overline{M}_{g,n}$ via st^*.

Another interesting map: $m \geq 0$

$$\pi_m : \overline{\mathcal{M}}_{g,n+m} \longrightarrow \mathcal{M}_{g,n}$$

forgetting the last m markings. (π_m is flat)

Q) For fixed d. is the pullback

$$\pi_m^* : R^d(\mathcal{M}_{g,n}) \longrightarrow R^d(\overline{\mathcal{M}}_{g,n+m})$$

injective for $m \gg 0$?