Intercontinental Moduli and Algebraic Geometry Zoominar

Counting surfaces on Calabi-Yau fourfolds

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[Part I. Introduction]

I.1 A leading question

- Let X be a smooth, projective Calabi-Yau fourfold / $\mathbb{C}$, i.e. $K_X \cong \mathcal{O}_X$ (includes HK, abelian fourfolds).
- $\gamma \in H^2(X, \Omega_X^2)$.

<u>Goal</u> Enumerate surfaces $S \hookrightarrow X$ in class γ via sheaf theory

<u>Two hurdles</u>

a/ A $(2,2)$ class γ does not remain $(2,2)$ as X deforms.

$\Rightarrow$ Deformation invariant quantities $= 0$

$$\underbrace{H^3(X, \Omega_X)}_{\text{obstruction space of a (2,2) class remains (2,2)}} \cong H^1(X, \Omega_X^3)^* \cong \underbrace{H^1(X, T_X)^*}_{1^{st} \text{ order deformation space of } X.}$$

(⚠ The Hodge conjecture is not known for CY fourfolds) first nontrivial case.

b/ Free roaming points and curves appear in the compactification.

We will resolve a/ in Part II and b/ in Part III.

Example of CY4: $X \subset \mathbb{P}^5$ degree 6 hypersurface.

$$h^{1,1} = 1, \quad h^{1,0} = h^{2,0} = h^{1,2} = 0, \quad h^{2,2} = 1752, \quad h^{1,3} = 426.$$

I.2 DT4 type virtual class (after Borisov-Joyce, Oh-Thomas)

$v = (0, 0, \gamma, \beta, n) \in \bigoplus H^{ev}(X, \mathbb{Q})$.

For simplicity, we consider the Hilbert scheme

$$I_v(X) = \{ I_Z \subset \mathcal{O}_X : ch_2(\mathcal{O}_Z) = \gamma,\ ch_3(\mathcal{O}_Z) = \beta,\ \chi(\mathcal{O}_Z) = n \}.$$

⊛ Below discussion generalizes to other moduli spaces.

· Deformation theory : $\mathrm{Def} = \mathrm{Ext}^1(I, I)_0$

$\mathrm{Obs} = \mathrm{Ext}^2(I, I)_0$ — Serre duality.

higher $= \mathrm{Ext}^3(I, I)_0$

Over the family,

$$\mathbb{I} \longrightarrow X \times I_v(X) \xrightarrow{\pi_X} X$$

i/ $\phi : \mathbb{E} = R\mathcal{H}om_{\pi_X}(\mathbb{I}, \mathbb{I})_0[3] \xrightarrow{At(\mathbb{I})} \mathbb{L}_{I_v(X)}$ [Huybrechts-Thomas]

ii/ $\theta : \mathbb{E} \xrightarrow{\sim} \mathbb{E}^\vee[2]$ (Serre duality)

iii/ $o : \mathcal{O}_I \xrightarrow{\sim} \det(\mathbb{E})$ s.t. $o^2 = \det(\theta)$ (Orientation).

Existence of o is due to [Cao-Gross-Joyce].

<u>Thm</u> [Borisov-Joyce, Oh-Thomas]. There exists a virtual class

$$[I_v(X)]^{vir} \in H_{2vd}(I_v(X), \mathbb{Z}) \quad \text{[BJ]}$$

$$\uparrow \mathrm{cl},\ [2^{-1}] \quad \text{[OT]}$$

$$[I_v(X)]^{vir} \in A_{vd}(I_v(X), \mathbb{Z}[\tfrac{1}{2}]) \quad \text{[OT]}$$

which depends on the choice of an orientation.
(different choice of o changes the sign of each connected component).

$$vd = n - \frac{1}{2}\gamma^2 \quad \leftarrow \text{does not depend on } ch_3.$$

- For special cases, earlier version [Cao-Leung]
- [BJ] integral class, [OT] $\hat{\mathcal{O}}^{vir}$, torus localization formula.

Local model (after [Brav-Bussi-Joyce], [Pantev-Toën-Vaquié-Vezzosi])

- U : smooth affine scheme, $\dim U = \mathrm{ext}^1(I,I)_0$
- $E \to U$: vector bundle of rank $= \mathrm{ext}^2(I,I)_0$
  $Q : E \otimes E \to \mathcal{O}_U$: nondegenerate symmetric bilinear form
- $t \in H^0(E)$: isotropic section ie $Q(t,t) = 0$.

$M := Z(s) \subset U$.

$$\begin{tikzcd}
T_U|_M \arrow[r, "(d\cdot t^*)^*"] & E|_M \overset{Q}{\cong} E^*|_M \arrow[r, "d\cdot t^*"] \arrow[d, "t^*"] & \Omega_U|_M \arrow[d, equal] & \mathbb{E} \arrow[d, "\phi"] \\
 & I_M/I_M^2 \arrow[r] & \Omega_U|_M & \mathbb{L}_M
\end{tikzcd}$$

If $\exists\, \Lambda \subset E$: maximal isotropic subbundle, t factors through Λ,

$$[M]^{vir} = \pm\, e(E,\Lambda,t) \cap [U] \in A_{vd}(M)$$ localized top Chern class.

General case : $\sqrt{e}(E,t)$. by [OT] using [Kiem-Li].

Counting points and curves on CY4

When $\gamma = 0$ (curves or points), the theory is related to GW invariants [Klemm-Pandharipande], [Cao-Maulik-Toda], [Cao-Kool], [Cao-Toda], ...

GW side

GW side is nice because $GW_{g,\beta} = 0$, $g \geq 2$.

[Part II. Reduction]

II.1 Variation of Hodge Structure (VHS)

$f: \mathcal{X} \longrightarrow (B, 0)$: smooth, projective morphism. $\mathcal{X}_0 \cong X$

B: smooth, connected, quasi-projective.

$\rightsquigarrow (\mathcal{H}^4, \mathcal{F}^p, \mathcal{H}^4_{\mathbb{Q}}, \nabla)$: VHS on B.

- $\mathcal{H}^4$: Hodge bundle. $\mathcal{H}^4|_b \cong H^4(\mathcal{X}_b, \mathbb{C})$. $b \in B$
- $\underset{\mathcal{F}^p}{\cup\!\mathsf{l}}$: filtration. $\mathcal{F}^p|_b \cong \bigoplus_{t \geq p} H^{4-t}(\mathcal{X}_b, \Omega^t_{\mathcal{X}_b})$.
- $\mathcal{H}^4_{\mathbb{Q}}$: rational structure $\mathcal{H}^4_{\mathbb{Q}}|_b \cong H^4(\mathcal{X}_b, \mathbb{Q})$
- $\nabla : \mathcal{H}^4 \longrightarrow \mathcal{H}^4 \otimes \Omega^1_B$: Gauss - Manin connection.

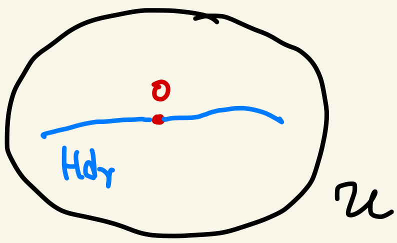
Hodge locus (analytic local description)

$\gamma \in H^{2,2}(\mathcal{X}_0, \mathbb{Q})$. Let $\mathcal{U} \subset B$: contractible nbh of 0.

$f|_{\mathcal{U}}$: topologically trivial $\tilde{\gamma} \in \Gamma(\mathcal{U}, \mathcal{H}^4_{\mathbb{Q}}) \cong H^4(X, \mathbb{Q})$.

Define the Hodge locus

$$Hd_\gamma(\mathcal{U}) = \{ b \in \mathcal{U} \mid \tilde{\gamma}(b) \in H^{2,2}(\mathcal{X}_b, \mathbb{Q}) \} \subseteq \mathcal{U}$$



For a general choice of γ, $\operatorname{codim}(Hd_\gamma, \mathcal{U}) > 0$ ($\Rightarrow$ virtual class $= 0$)

Curves on surfaces

This locus is called the Noether - Lefschetz locus. Related reduction in the enumerative geometry is developed by

[Li] [Bryan-Leung], [Maulik-Pandharipande-Thomas],

[Kiem-Li], [Kool-Thomas], $\cdots$

II.2 Deformation of sheaves vs Hodge classes

X : smooth. projective CY4, $\omega \in H^0(\Omega_X^4)$: holo. volume form

Define a symmetric bilinear form

$$B_\gamma : H^1(T_X) \otimes H^1(T_X) \to \mathbb{C}, \quad \xi_1 \otimes \xi_2 \mapsto \int_X \lrcorner_{\xi_2} \lrcorner_{\xi_1}(\gamma) \cup \omega$$

where $\lrcorner : H^1(T_X) \otimes H^q(\Omega_X^p) \to H^{q+1}(\Omega_X^{p-1})$ is a contraction

Let $\rho_\gamma = \mathrm{rank}(B_\gamma)$ $(\leq h^{1,3}(X))$.

The key diagram (due to Bloch, Buchweitz-Flenner)

Let I : sheaf (or complex) on X with $ch_1(I) = 0$, $ch_2(I) = \gamma$. Then

$$\begin{tikzcd}
H^1(T_X) \arrow[rr, "ob"] \arrow[dr, "\lrcorner.(\gamma)"'] \arrow[ddr, bend right, "B_\gamma"'] & {} \arrow[d, phantom, "\circlearrowright"] & \mathrm{Ext}^2(I, I)_0 \arrow[dl, "SR"] \arrow[ddl, bend left, "ob^*"] \\
 & H^3(\Omega_X) \arrow[d, "\cong", "SD"'] & \\
 & H^1(T_X)^* &
\end{tikzcd}$$

- $At(I) \in \mathrm{Ext}^1(I, I \otimes \Omega_X)$ Atiyah class
- $ob(\xi) = \lrcorner_\xi(At(I))$ ← Sheaf theoretic obstruction
- $SR(-) = tr((-) \cup At(I))$ Semiregular map
- $\lrcorner_-(\gamma)$ = contraction (IVHS) ← Hodge theoretic obstruction

related by SR.

Special feature of CY4 and (2,2) class

$$tr(ob(\xi_1) \cup ob(\xi_2)) = B_\gamma(\xi_1, \xi_2)$$

$\Rightarrow$ ob preserves symmetric bilinear forms. and $SR = ob^*$.

II.3 Reduced obstruction theory

Recall : $v = (0,0,\gamma,*,*)$, $I_v(X)$: Hilbert scheme,

$\phi : \mathbb{E} = R\mathcal{H}om_\pi(\mathbb{I},\mathbb{I})_0[3] \xrightarrow{At(\mathbb{I})} \mathbb{L}_{I_v(X)}$: obstruction theory.

Surjective cosections (removing trivial components from the obs space).

Over the moduli space, the semi-regularity map induces a map

$$SR : \mathbb{E}[-1] \longrightarrow H^1(T_X)^* \otimes \mathcal{O}_{I_v(X)}, \qquad SR^2 = B_\gamma \otimes 1$$

Choose a maximal nondegenerate subspace $V \subseteq H^1(T_X)$.

$$\begin{tikzcd} V \arrow[r, hook] \arrow[rr, bend right, "\cong"'] & H^1(T_X) \arrow[r, two heads] & H^1(T_X)/\ker(B_\gamma) := H^1(T_X)_\gamma . \end{tikzcd}$$

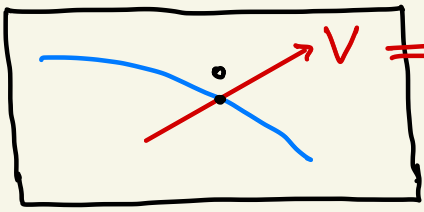
↑ Choose orientation

Then

$$SR_V : \mathbb{E}[-1] \xrightarrow{SR} H^1(T_X)^* \otimes \mathcal{O}_{I_v(X)} \longrightarrow\!\!\!\!\to V^* \otimes \mathcal{O}_{I_v(X)}$$

$\rightsquigarrow \mathbb{E} \cong (\mathbb{E}^{red}) \oplus (V \otimes \mathcal{O}[1]) \qquad \mathbb{E}^{red} = \mathrm{Cone}(SR_V^\vee[+1])$.

$$\begin{array}{c} \mathcal{X} \\ \downarrow f \\ \mathcal{U} \subset B \end{array}$$



V = a transverse slice to the Hodge locus

$T_0\mathcal{U} \underset{KS}{\cong} H^1(T_X)$.

The main theorem

Thm (BKP) There exists a reduced virtual class

$$[I_v(X)]^{red} \in A_{rvd}(I_v(X)), \quad rvd = n + \tfrac{1}{2}(P_\gamma - \gamma^2)$$

depending on a choice of orientation on $\mathbb{E}$ & $H^1(T_X)_\gamma$. Moreover, there exists a reduced virtual structure sheaf $\hat{\mathcal{O}}^{red} \in K_0(I_v(X))$.

- The class is independent of the choice of $V \subset H^1(T_X)$.
- Proof uses [Kiem-Li], [Kiem-Park].
- We further checked that $\mathbb{E}^{red}$ is a reduced obstruction theory adopting the algebraic twistor method of [Kool-Thomas].

II.4 Examples

Ideal geometry

<u>Def</u> A point $[I] \in I_v(X)$ is semi-regular if

$$SR: \mathrm{Ext}^2(I,I)_0 \longrightarrow H^3(\Omega_X)$$ is injective. (ie ob is surjective)

<u>Thm</u> (Bloch, BKP) A point $[I]$ is semiregular if and only if $I_v(X)$ is smooth of dim = rvd at $[I]$.

- If γ is represented by semi-regular sheaf $\rho_\gamma - \gamma^2$ = even.
- Reduced theory can be though of as a tool to handle non semi-regular situations.

Comments on \rho

a/ If the Kodaira-Spencer map is an isomorphism,

ρ_γ = codim of Hdg_γ, when it is generically reduced.

b/ $0 \le \rho_\gamma \le h^{1,3}$. Usually ρ_γ is very far from $h^{1,3}$

c/ If $H^{2,0}(X) = 0$ and $\gamma = D_1 \cdot D_2$, where $D_1, D_2 \in H^{1,1}(X)$.

Then $\rho_\gamma = 0$ and we have interesting non-reduced invariants.

Some values of \rho

i/ $X = V(f) \subset \mathbb{P}^5$. $\Rightarrow$ Jacobian ring of f encodes IVHS of X.

$\rightsquigarrow$ values of ρ_γ can be computed via period integrals.

e.g. X = Fermat sextic. HC is known. ρ_γ computed for each γ.

$\mathbb{P}^2 \subset X$ of type (1,1,1). $\rho_\gamma = 19$. $\mathrm{rvd} = 1 + \frac{1}{2}(19-21) = 0$

ii/ $S \hookrightarrow X$: local complete intersection. $\left(\mathrm{vd} = 1 - 21/2 = -\frac{19}{2}\right)$

$$\begin{tikzcd} H^1(T_X) \arrow[r, "\alpha"] \arrow[dr, "ob"'] & H^1(N_{S/X}) \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Ext}^2(I_{S/X}, I_{S/X}) \end{tikzcd}$$

$\rightsquigarrow \rho_\gamma \le \dim H^1(N_{S/X})$

equals when α : surjective.

e.g. $\mathbb{P}^1 \times \mathbb{P}^1 \subset K3 \times K3$. $\Rightarrow$ $\rho_\gamma = 2$, $\mathrm{rvd} = 1 + \frac{1}{2}(2-4) = 0$

II.4 Deformation invariance

$f: \mathcal{X} \longrightarrow (B, 0)$: smooth, projective morphism. $\mathcal{X}_0 \cong X$

B : smooth, connected, quasi-projective s.t

$\omega_{\mathcal{X}/B} \cong \mathcal{O}_{\mathcal{X}}$ (true Zariski locally on B).

Choose a $\mathbb{Q}$-section $\tilde{v} \in \bigoplus \mathcal{H}^{\cdot}_{\mathbb{Q}}(\mathcal{X}/B)$ with $\tilde{v}(0) = v$.

$$I_{\tilde{v}}(\mathcal{X}/B) \longrightarrow B$$

Global invariance cycle Thm (Deligne) $\tilde{v}_i(b)$ is pure (i,i) class.

$\rightsquigarrow B$ lies in the Hodge locus.

Thm (BKP). Suppose there exists a family of orientations on $I_{\tilde{v}}(\mathcal{X}/B)/B$.

If $\rho(\tilde{v}_2(b))$ is constant $(*)$ on B, then there exists a reduced class

$$[I_{\tilde{v}}(\mathcal{X}/B)]^{red} \in A_{rvd + \dim B}(I_{\tilde{v}}(\mathcal{X}/B)) \quad \text{s.t}$$

$$\begin{tikzcd} \mathcal{X}_b \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ b \arrow[r, hook, "i_b"] & B, \end{tikzcd} \qquad i_b^![I_{\tilde{v}}(\mathcal{X}/B)]^{red} = [I_{\tilde{v}(b)}(\mathcal{X}_b)]^{red}$$

- Works also for $\hat{\mathcal{O}}^{red}$.
- If we choose a coh/K-theoretic insertion on the family, we get deformation invariance of numbers over the base

Remark We expect there could be an obstruction of the existence of family orientation on $I_{\tilde{v}}(\mathcal{X}/B)$ in general. At least up to finite étale cover of B, there exists a family orientation.

Remark Condition $(*)$ is not restrictive.

Variational Hodge Conjecture (VHC)

Conjecture (Grothendieck) $f: X \to B$: smooth, projective, B: smooth, connected, quasi-projective. $v \in H^0(B, R^{2p}f_* \mathbb{Q})$. If $v(0)$ is a cohomology class of an algebraic cycle at some $0 \in B(\mathbb{C})$, then the same holds for all $v(b)$, $b \in B(\mathbb{C})$.

Note · Hodge Conjecture $\Rightarrow$ VHC

· Not much is known for VHC.

technical

Thm (BKP) Let $f: X \to B$: family of CY4. Assume $H^2(\Theta_X) = 0$. If $[I_{\tilde{v}(b)}(X_b)]^{red} \neq 0$ in A_* for some $b \in B$, then VHC holds for $\tilde{v}_2(b)$.

Remarks a/ We do not assume the existence of family orientation or $\rho(\gamma)$ constant (maximal).

b/ This recovers the result of [Bloch], [Buchweitz - Flenner] (semi-regular $\Rightarrow$ VHC).

Examples i/ $X \subset \mathbb{P}^5$, $S \subset X$: complete intersection.

(Steenbrink) $I_{S/X}$ is semi-regular. They are rigid when

(1.1.1) (1.1.2) (1.1.3) (1.2.2)

(1.2.3) (2.2.2) (2.2.3) (1.1.4)

ii/ any "low degree" (Fano) surface $S \subset$ Complete Intersection CY4.

[Part III. Moduli]

III.1 Moduli of pairs

- X : smooth projective variety $\dim X \geq 4$.

Hilbert schemes are too big because of free roaming points, curves.

We consider alternative compactifications following [Pandharipande-Thomas]

$\mathrm{Pair}(X, v) = \{ \mathcal{O}_X \xrightarrow{s} F : ch(F) = v \}$ moduli stack of pairs. $\dim F = 2$.

$T_0(F) \subset T_1(F) \subset F$: torsion filtration. $Q := \mathrm{coker}(s)$.

Def ① (F, s) is DT(=PT-1) stable if s is surjective ($F \cong \mathcal{O}_Z$)

② (F, s) is PT_0 stable if $T_0(F) = 0$, $\dim Q \leq 0$.

③ (F, s) is PT_1 stable if $T_1(F) = 0$, $\dim Q \leq 1$ (F: pure 2dim)

$$P_v^{(q)}(X) = \{ (F, s) : PT_q \text{ stable}, ch(F) = v \} \overset{\text{open}}{\subseteq} \mathrm{Pair}(X, v).$$

Construction of moduli spaces

Thm (BKP) There exists a projective scheme N, reductive group $G \curvearrowright N$, G-linearization $\mathcal{L}_{-1}, \mathcal{L}_0, \mathcal{L}_1$ such that, there exists no strictly semistable and

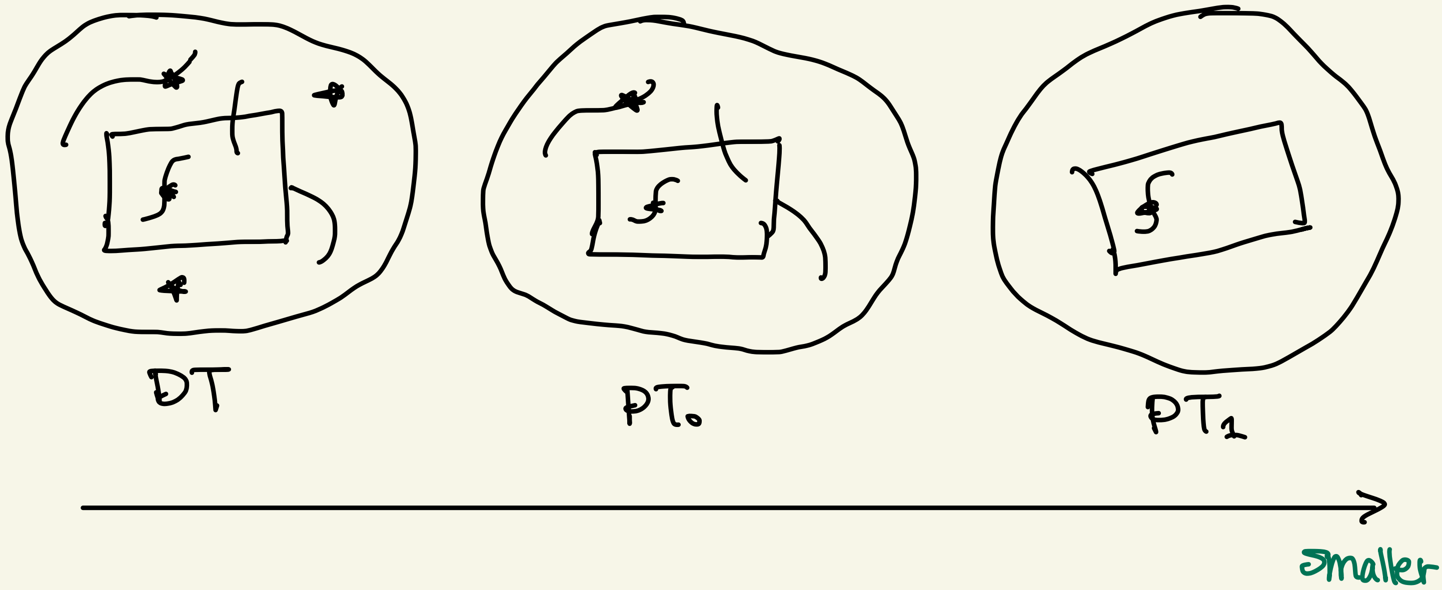
$$N /\!/_{\mathcal{L}_q} G \cong P_v^{(q)}(X) \qquad q = -1, 0, 1$$

Hence $P_v^{(q)}(X)$ are projective and connected by GIT wall-crossing.

Remark a/ Existence of DT (Grothendieck), PT_1 (Le Potier) is known. Moduli spaces of non-pure sheaves is not much studied. The construction of PT_0 seems new.

b/ Our construction is motivated by [Stoppa-Thomas]

Pointwise description



a/ PT_1 pair. If $S = \mathrm{Supp}(F)$ is smooth, $F \cong I_{Z\subset S}(C)$ where

$$\underset{\text{0-dim.}}{Z} \hookrightarrow \underset{\text{pure 1-dim'l}}{C} \hookrightarrow S$$

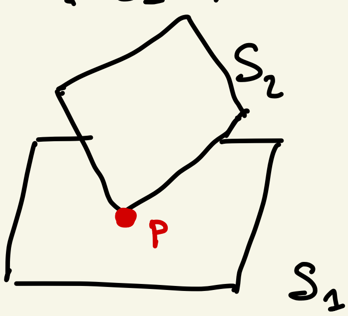
b/ PT_0 pair. If $S = \mathrm{Supp}(F)^{\text{pure}}$ is Cohen-Macaulay,

$$\begin{tikzcd} 0 \arrow[r] & I_{S/X} \arrow[r] \arrow[d, dotted, "\bar{s}"] & \mathcal{O}_X \arrow[r] \arrow[d, "s"] & \mathcal{O}_S \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T_1(F) \arrow[r] & F \arrow[r] & F/T_1(F) \arrow[r] & 0 \end{tikzcd}$$

$(F, s) \Leftrightarrow (I_{S/X}, \bar{s})$.

c/ $PT_0 = PT_1$ pair. If S is not CM, then there exist $PT_0 (= PT_1)$ but not DT

e.g. $S_1, S_2 \subset X$ smooth surfaces. $S_1 \cap S_2 = p$, $F = \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2}$



$$\mathcal{O}_X \xrightarrow{s} \mathcal{O}_{S_1} \oplus \mathcal{O}_{S_2} \longrightarrow \mathcal{O}_p \longrightarrow 0$$

III.2 Pairs as complexes

Powerful idea of [Thomas] [PT]:

$P_v^{(q)}(X)$ is a fine moduli space. There exists universal sheaf with section

$$\mathbb{I}^\bullet := [\mathcal{O}_{X\times P} \xrightarrow{s} \mathbb{F}] \quad \text{on } X \times P_v^{(q)}(X).$$

We consider (F,s) as an element in $D^b(X)$:

$$P_v^{(q)}(X) \xrightarrow{\mathbb{I}^\bullet} \mathrm{Perf}(X)^{spl}_{\mathcal{O}_X} \leftarrow \text{(-2)-shifted str.}$$

moduli stack of perfect complex E
$\det(E) \cong \mathcal{O}_X$, $\mathrm{Ext}^{<0} = 0$ $\mathrm{Ext}^0 = \mathbb{C}$.

Thm (BKP) The above morphism is an open immersion.

$\Rightarrow$ $\exists$ 3 term symmetric obstruction theory on $P_v^{(q)}(X)$ and hence
$\exists$ reduced cycle

Remark PT_q stability conditions can be realized by Bayer's polynomial stab.

Remark When $q = 1$, the statement is independently proven by [Gholampour–Jiang–Lo].

Questions

(A) Correspondence between DT/PT_0 invariants?

(B) Correspondence between PT_0/PT_1 invariants?

(C) Structure of PT_1 invariants?

[Part IV. Invariants]

IV.1 DT/PT0 correspondence

K-theoretic insertion

$$\pi_X, \pi_P : X \times P_v^{(q)}(X) \rightrightarrows X,\ P_v^{(q)}(X), \quad \mathbb{F} \to X \times P_v^{(q)}(X).$$

For $L \in \mathrm{Pic}(X)$, $L^{[v]} := R\pi_{P*}(\mathbb{F} \otimes \pi_X^* L) \in K^0(P_v^{(q)}(X))$

Let y be a formal variable. The Nekrasov genus is given by

$$\langle\langle L \rangle\rangle_{X,v}^{(q)} = \chi\left(P_v^{(q)}(X),\ \hat{\mathcal{O}}^{red/vir} \otimes \hat{\Lambda}(L^{[v]} \otimes y^{-1})\right) \text{ where}$$

$$\hat{\Lambda}(E) := \sum_i (-1)^i \frac{\Lambda^i E}{\sqrt{\det E}}, \quad E \in K^0(P_v^{(q)}(X))$$

Conjectural DT/PT0 correspondence

Conjecture (BKP) Fix $\gamma \cdot \beta$. Then there exists a choice of orientations s.t

$$\frac{\sum_n \langle\langle L \rangle\rangle^{DT}_{X,\gamma,\beta,n}\, q^n}{\sum_n \langle\langle L \rangle\rangle^{DT}_{X,0,0,n}\, q^n} = \sum_n \langle\langle L \rangle\rangle^{PT_0}_{X,\gamma,\beta,n}\, q^n$$

Remark a/ Taking $y \to 1^+$, we get cohomological DT/PT0.

b/ When $\gamma = 0$, the conjecture appears in [Cao-Kool-Monavari].

c/ The cohomological version of the denominator, $L = \mathcal{O}(D)$. [Park]

d/ Assuming [Joyce], the closed formula of the denominator is given by the plethystic exponential [Bojko].

e/ Other insertions (ie $rk(\alpha) \geq 2$) do not work.

This conjecture is motivated from toric computations (non-reduced)

IV.2 Toric Calabi-Yau fourfold

X : smooth toric CY4. $T = (\mathbb{C}^*)^3 \curvearrowright X$ Calabi-Yau torus
$\{t_1t_2t_3t_4 = 1\}$

Vertex formalism

On each toric chart $X \cong \mathbb{C}^4$, we fix $Z \subset X$: 2-dim, T-inv, no embedded points.
(asymptotic behavior)

$\mathcal{O}_Z|_{(\mathbb{C}^*)^2 \times \mathbb{C}^2} \cong \mathcal{O}_A[X_1^{\pm}, X_2^{\pm}]$, $A \subset \mathbb{C}^2$: 0-dim $\rightsquigarrow \{\lambda_{ij}\}_{1\le i<j\le 4}$ 2D part.

$\mathcal{O}_Z|_{\mathbb{C}^* \times \mathbb{C}^3} \cong \mathcal{O}_B[X_1^{\pm}]$, $B \subset \mathbb{C}^3$: 1-dim $\rightsquigarrow \{\mu_i\}_{1\le i\le 4}$ 3D part.

<u>Thm</u> (BKP) Fix λ, μ s.t. there is no moduli on PT_0 side (ie T-fixed are 0-dim, reduced). Then there exist topological vertices (K-theoretic)

$$V^{DT}_{\lambda,\mu}(q,t,y), \quad V^{PT_0}_{\lambda,\mu}(q,t,y)$$

which depend on the choice of signs.

Equivariant correspondence

<u>Conjecture'</u> Under the same assumption, there exists a choice of signs st

$$\frac{V^{DT}_{\lambda,\mu}(q,t,y)}{V^{DT}_{\phi,\phi}(q,t,y)} = V^{PT_0}_{\lambda,\mu}(q,t,y).$$

Up to sign issues, Conjecture' $\Rightarrow$ equiv. version of Conjecture by localization [OT].

<u>Remark</u> a/ Finding a nice formula on T-fixed locus is a real challenge.
b/ T-fixed on DT is always reduced point.
T-fixed on PT^0 can be singular
c/ The denominator has a closed formula [Nekrasov-Piazzalunga] proven by [Kool-Rennemo]

Examples

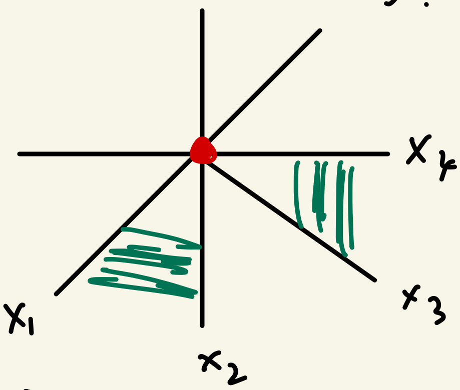
- Crossed instantons ← due to [Nekrasov, gauge origami]

This consider the case when $PT_0 = PT_1$ ie λ general. $\mu = \mu_{min}$.

Lemma $S \subset X$: pure 2-dim'l subscheme.

- S is CM $\iff$ $\mathcal{E}xt^3_X(\mathcal{O}_S, \mathcal{O}_X) = 0$
- If (F, s) is supported on S is both $PT_0 = PT_1$, then
$$\ell(\mathrm{Coker}(s)) \le \ell(\mathcal{E}xt^3_X(\mathcal{O}_S, \mathcal{O}_X)).$$

Eg. Take $S = \mathbb{C}^2 \cup_{pt} \mathbb{C}^2$.



Then the (normalized) topological vertex is

$$V^{PT_0}_{(1),\phi,\phi,\phi,\phi,(1)\ \mu_{min}} = 1 + \frac{[t_1t_2][y]}{[t_1t_3][t_2t_3]}\, q$$

$[x] := x^{\frac{1}{2}} - x^{-\frac{1}{2}}$.

Final remarks

a) For special geometry, $L \cong \mathcal{O}_X(D)$. $D \hookrightarrow X$: smooth divisor, γ pushed from D, we checked Conjecture on compact geometry.

b) (work in progress B-Bojko-Lim). Assuming Joyce's wall-crossing formula, Conjecture holds for X = strict CY4, non-reduced.

IV.3 PT0/PT1 correspondence

* We expect PT_0 and PT_1 invariants are related by curve counting invariants. We don't know the correspondence in full generality. We focus on special geometry.

Weierstrass Calabi-Yau fourfolds

B : smooth, projective Fano 3fold. (or CY3).

Weierstrass CY4 :

$$\begin{tikzcd} & X \arrow[r, hook] \arrow[d, "p"'] & \mathbb{P}(\mathcal{O}\oplus L^2\oplus L^3) \arrow[dl] \\ \Delta \arrow[r, hook] & B & \end{tikzcd}$$

$L = K_B$.

· general fiber of p : smooth elliptic curve

· Singular fiber : nodal, cusp

<u>Example</u> $B=\mathbb{P}^3$. $h^{1,1}=2$, $h^{1,0}=h^{2,0}=h^{1,2}=0$, $h^{2,2}=15{,}564$, $h^{1,3}=3878$.

Let $f\in H^6(X,\mathbb{Z})$ be a fiber class.

<u>Conjecture</u> (BKP) For $\beta\in H_2(B,\mathbb{Z})$, $\gamma=p^*\beta$, $n=\dfrac{\beta\cdot\Delta}{12}$. Then there exists a choice of orientations such that

$$\frac{\sum Q^d\int_{[P^{(0)}_{\gamma,df,n}]^{vir}}\prod \tau_{k_i}(\sigma_i)}{\sum Q^d\int_{[P_{0,df,0}]^{vir}} 1} = \sum Q^d\int_{[P^{(1)}_{\gamma,df,n}(X)]^{vir}}\prod \tau_{k_i}(\sigma_i).$$

for all $\sigma_i\in H^*(X)$ with $p_*(\sigma_i)\in H^{\geq 2}(B)$.

$\tau_{k_i}(\sigma_i)$ = k-th descendent insertion

<u>Thm</u> (BKP) Conjecture is true when all pure $C\subset B$, $[C]=\beta$ is irreducible Gorenstein and

i/ B : toric Fano 3fold : via stationary DT/PT on B [Oblomkov – Okounkov – Pandharipande]

ii/ B : CY3 : DT/PT on B [Bridgeland] [Toda].

<u>Question</u> What about a K theoretic correspondence?

Moving one irreducible curve

$v = (0, 0, \gamma, \beta, n) \in \bigoplus H^*(X, \mathbb{Q})$.

Assume i/ $\forall$ pure surface $S \subset X$ with $[S] = \gamma$ are CM with the constant $ch(\mathcal{O}_S) =: v(\gamma)$

ii/ $\beta - v_3(\gamma)$ is irreducible, effective.

iii/ $\forall$ pure 1-dimensional sheaf G on X with $ch(G) = v - v(\gamma)$,

$$\mathrm{Ext}^2(I_{S/X}, G) = 0, \quad \forall [I_{S/X}] \in I_{v(\gamma)}(X).$$

Consider a morphism

moduli of 1-dim'l stable sheaves.

$$\Phi : P_v^{(0)}(X) \longrightarrow M_{v - v(\gamma)}(X) \times I_{v(\gamma)}(X)$$
$$(F, s) \longmapsto (T_1(F), I_{\mathrm{supp}(F)/X}).$$

Thm (BKP) Under the above assumption, Φ is virtually smooth and

$$[P_v^{(0)}(X)]^{red} = \Phi^! \left([M_{v-v(\gamma)}(X)]^{vir} \times [I_{v(\gamma)}(X)]^{red} \right)$$

virtual pullback [Manolache]

Remark a/ Assumption i/ implies $v(\gamma)$ is "minimal". This is the case when $DT = PT_0 = PT_1$

b/ The proof involves functorial property of Oh-Thomas class developed by [Park]

c/ Φ is a virtual projective bundle and $\exists$ pushforward formula along Φ.

IV.4 Structure of PT1 invariants

Rigid surface inside CY4

$i: S \hookrightarrow X$ smooth projective surface. $N := N_{S|X}$. $\Rightarrow \det(N) \cong K_S$.

Assume i/ $[S] = \gamma$ is irreducible
ii/ $H^0(S, N) = 0$ (ie $S \hookrightarrow X$ cannot move)
iii/ $I_{S|X}$ is semiregular.

For $\beta \in H^2(S, \mathbb{Z})$ effective, consider the relative Hilbert scheme

$$S_\beta^{[m]} = \{ Z \subset C \subset S : [C] = \beta, \chi(\mathcal{O}_Z) = m \} \xhookrightarrow{j} S_\beta \times S^{[m]}$$

It has a natural pair obstruction theory by [Kool], [Kool-Thomas].

Thm (BKP) Under the above assumption, the pushforward i_* induces an open immersion

$$\Phi : S_\beta^{[m]} \hookrightarrow P_v^{(1)}(X). \qquad v = ch(i_* I_{Z|S}(C)).$$

Moreover, $\Phi^* [P_v^{(1)}(X)]^{red} = \pm a_m j^*(pt) \cap [S_\beta^{[m]}]^{vir}$, where

$$\prod_n (1-q^n)^{c_2(N) - e(S)} =: \sum a_m q^m, \qquad pt \in H_0(S^{[m]}).$$

The additional obstruction bundle can be computed by [Carlson-Okounkov].

Pair/Sheaf correspondence

Assume γ is irreducible. $v = (0, 0, \gamma, *, *) \in \oplus H^*(X)$.
Consider the moduli space of two dimensional torsion stable sheaves.

$$M_v(X) = \{ F : F : \text{stable}, ch(F) = v \}.$$

(independent of the choice of polarization because γ = irred)

$$[M_v(X)]^{red} \in A_{1 + \frac{1}{2}(\beta_\gamma - \gamma^2)}(M_v(X))$$

rigidified obs theory

Let

$$\Phi : P_v^{(1)}(X) \longrightarrow M_v(X), \quad (F, s) \longmapsto F$$

Assume i/ $H^2(X, F) = 0 \quad \forall [F] \in M_v(X)$ ← crucial.
ii/ $\exists$ universal sheaf $\mathbb{G}$ on $X \times M_v(X)$

Thm (BKP) Under the above assumption,

$$\Phi^! [M_v(X)]^{red} = [P_v^{(1)}(X)]^{red}.$$

Generating series (?)

Full picture is not clear.

Lemma For fixed $\gamma.\beta$, $\exists N>0$. s.t $P^{(1)}_{\gamma.\beta.n}(X)=\phi$, $n\geq N$. Moreover if $\chi(F)$ is maximal, F is a reflexive sheaf.

eg If $S=\mathrm{Supp}\,F$ is smooth, $F=I_{Z|S}(C)$. $\chi(F)$ is maximal when $l(\mathcal{O}_Z)=0$. In that case $F=\mathcal{O}_S(C)$.

Case 1. Fix $\gamma.\beta$ and sum over n; finitely many terms

Case 2. Fix γ and sum over β : This seems an interesting direction (eg. Weierstrass model). We should not fix n, but rather we should take $n=n_{max}(\gamma.\beta)$. (reflexive case)

- Natural cohomological insertion? or K theoretic inv?

Case 3. Sum over γ : Most complicated question.

The $\rho(Gr)$ is not additive nor constant. rvd jumps around

Many other questions remain ... Relation to string theory? ...