Ngô's support Theorem

§1. Weak abelian fibration

k : finite field $\quad \ell$: prime $\neq \mathrm{char}(k)$.

<u>Def</u> A <u>weak abelian fibration</u> is (M, P, S)

$$\begin{tikzcd} P \arrow[r, phantom, "\curvearrowright"] \arrow[dr, "g"'] & M \arrow[d, "f"] \\ & S \end{tikzcd}$$

- f : proper
- g : sm com gp sch (conn. fib) $\quad$ s.t.
- act : $P \times_S M \longrightarrow M$

a/ f, g : rel dim $= d$ (pure)

b/ $\forall$ geom pt $m \in M$. $P_s \curvearrowright M_s$ affine st

c/ $T_{\overline{\mathbb{Q}}_\ell}(P)$ is polarizable ($\Leftarrow g$: q.proj)

By Chevalley's Thm. $\forall s \in S$.

$$1 \longrightarrow R_s^{\mathrm{aff}} \longrightarrow P_s \longrightarrow A_s^{ab} \longrightarrow 1.$$

$\delta : S \to \mathbb{Z}_{\geq 0} \quad s \longmapsto \dim(R_s^{\mathrm{aff}})$ $\quad$ Constr. (upper semi-continuous)

<u>Def</u>: $P \to S$ is <u>δ-regular</u> if $\mathrm{codim}\{s \in S : \delta(s) \geq \delta\} \geq \delta$ $\quad \forall \delta \geq 0$

↳ Not appear in this talk.

§2 Main result:

<u>Thm</u> (Support). Suppose (M, P, S) : w.af/k. Assume M : sm ($\Rightarrow \mathbb{D}(\overline{\mathbb{Q}}_\ell) \cong \overline{\mathbb{Q}}_\ell[2n](n)$, $n = \dim M$). Let K : simple perv sh appear in $f_* \overline{\mathbb{Q}}_\ell$. Z : supp of $K \hookrightarrow S$. Then

$$\mathrm{codim}\, Z \leq \delta_Z.$$

If "=", $\exists\, U \hookrightarrow S$, $U \cap Z \neq \phi$ & nontriv. loc sys L on $U \cap Z$ s.t. L is a direct factor of $R^{2d} f_* \overline{\mathbb{Q}}_\ell|_U$.

<u>Rmk</u> ① f has integral fibers $\Rightarrow$ full support.

② [Maulik-Shen] : <u>generalized</u> to $f_* IC_M$.

<u>Thm</u> (Goresky-MacPherson). $f : M \to S$. proper. rel dim = pure d. M : sm. Then $\mathrm{codim}(Z) \leq d$.

Pf) $\mathrm{occ}(Z) = \{i \in \mathbb{Z} : {}^pH^i(f_* \overline{\mathbb{Q}}_\ell) \neq 0\} \neq \phi$

${}^pH^i(\mathbb{D}K) \cong \mathbb{D}({}^pH^{-i}K) \Rightarrow \mathrm{occ}(Z)$ is sym along $\dim M$

$\Rightarrow \exists\, n \in \mathrm{occ}(Z)$, $n \geq \dim M$. So, $U \hookrightarrow S$. L : loc sys $U \cap Z$ s.t. $(L[\dim Z])[-n]$ is a direct comp of $f_* \overline{\mathbb{Q}}_\ell|_U$.

$\Rightarrow H^{n - \dim Z}(f_* \overline{\mathbb{Q}}_\ell) \neq 0 \Rightarrow \dim M - \dim Z \leq n - \dim Z \leq 2d$

$\blacksquare$

Heuristic idea: $s \in Z$. Suppose ét loc $s \in S'$.

- A_s lifts to $A_{S'} \to S'$ ab. sch.
- $\exists\, A_{S'} \to P_{S'}$ split. upto isogeny.

$\Rightarrow$ $A_{S'} \curvearrowright M_{S'}$ finite stabilizer.

$$M_{S'} \xrightarrow{\text{sm. prop}} M_{S'}/A_{S'} \xrightarrow{\text{prop}} * \quad \Rightarrow \text{Leray} + \text{G-M}.$$

$\uparrow$ rel dim $= \delta_Z$

This never happens! Instead, we linearize the problem!

§3 Tate module. ← Linearize group & action.

$g : P \to B$ sm. comm. gp. rel. dim $= d$

$$\Lambda_P := g_! \overline{\mathbb{Q}}_\ell[2d](d)$$

<u>Def</u> (Tate module) $T_{\overline{\mathbb{Q}}_\ell}(P) := H^{-1}(\Lambda_P)$

<u>Ex</u> A_s : ab $\Rightarrow$ $T_{\overline{\mathbb{Q}}_\ell}(A_s) = H^{2d-1}(A_s, \overline{\mathbb{Q}}_\ell)(d)$ pure wt $= -1$

$(\mathbb{G}_m)^d$: $\Rightarrow$ $T_{\overline{\mathbb{Q}}_\ell}(\mathbb{G}_m^d) = H_c^{2d-1}(\mathbb{G}_m^d, \overline{\mathbb{Q}}_\ell)(d)$ pure wt $= -2$.

$N \in \mathbb{Z}$. $[N]^* \curvearrowright \Lambda_P$ $\Rightarrow$ $\exists$ eigen sp decomp

$$\Lambda_P = \bigoplus_{i \geq 0} \wedge^i T_{\overline{\mathbb{Q}}_\ell}(P)[i]$$

<u>Def</u> $T_{\overline{\mathbb{Q}}_\ell}(P)$ is <u>polarizable</u> if ét loc S. $\exists$ altern. bilinear

$$T_{\overline{\mathbb{Q}}_\ell}(P) \times T_{\overline{\mathbb{Q}}_\ell}(P) \longrightarrow \overline{\mathbb{Q}}_\ell$$

s.t. $\forall\, s \in S$. induces a non-deg on $T_{\overline{\mathbb{Q}}_\ell}(A)$.

act : $P \times_S M \to M$. : sm. By purity isom :

$$\overline{\mathbb{Q}}_\ell[2d](d) \xrightarrow{\sim} \mathrm{act}^! \overline{\mathbb{Q}}_\ell$$

$\rightsquigarrow$ tr : $\mathrm{act}_! \overline{\mathbb{Q}}_\ell[2d](d) \longrightarrow \overline{\mathbb{Q}}_\ell$

$$\begin{array}{ccc} P \times_S M & \xrightarrow{\text{act}} & M \\ & \searrow^{g \times_S f} \quad \circlearrowleft & \downarrow f \\ & & S \end{array}$$

Take $f_! = f_*$

Künneth : $\Lambda_P \otimes f_* \overline{\mathbb{Q}}_\ell \longrightarrow f_* \overline{\mathbb{Q}}_\ell$

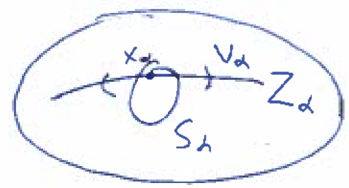
<u>Goal</u> : $f_* \overline{\mathbb{Q}}_\ell$ is <u>free over abelian part</u> of Λ_P.

§4 Decomposition theorem :

$$f_* \overline{\mathbb{Q}}_\ell \cong \bigoplus_n {}^pH^n(f_* \overline{\mathbb{Q}}_\ell)[-n] \quad \text{on } S_{\bar{k}}.$$

- ${}^pH^n(f_* \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{\alpha \in I} K_\alpha^n$ $\quad I =$ Index set for supports.
- $K_\alpha^n = \bigoplus$ simple. supported on $Z_\alpha \hookrightarrow S_{\bar{k}}$
- $K_\alpha := \bigoplus_n K_\alpha^n[-n]$ ← On V_α, $K_\alpha^n|_{V_\alpha} = \mathcal{L}_\alpha^n[\dim V_\alpha]$

$\uparrow$ pure loc sys of wt $= n$.



$S_{\bar{k}}$

S_α = strict Henselization of $S_{\bar{k}}$ at x_α.

$\Rightarrow$ $K_\alpha|_{S_\alpha}$ = graded v. sp at x_α

$$1 \to R_\alpha \to P_\alpha = P_{x_\alpha} \longrightarrow A_\alpha \longrightarrow 1$$

§5. Freeness

$$\Lambda_P \otimes f_*\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell$$

$$\rightsquigarrow T_{\overline{\mathbb{Q}}_\ell}(P) \otimes {}^pH^n(f_*\overline{\mathbb{Q}}_\ell) \to {}^pH^{n-1}(f_*\overline{\mathbb{Q}}_\ell)$$

$$\overset{wt}{\rightsquigarrow} T_{\overline{\mathbb{Q}}_\ell}(A_\alpha) \otimes K^n_{\alpha, x_\alpha} \to K^{n-1}_{\alpha, x_\alpha}$$

<u>Prop</u> (Freeness) K^n_{α, x_α} is a free graded Λ_{A_α}-module

<u>Cor</u> Freeness $\Rightarrow$ Supp Thm

Pf) Similar to G-M. $occ(Z_\alpha) := \{i \in \mathbb{Z} : K^i_{\alpha, x_\alpha} \neq 0\}$

Freeness $\Rightarrow occ(Z_\alpha) = \bigcup$ Intervals of length $2(d-\delta_\alpha)$

PD $\Rightarrow \exists n \in occ(Z_\alpha) : n \geq \dim M + d - \delta_\alpha$

$H^{n-\dim Z_\alpha}(K^n_\alpha[-n]) \neq 0 \Rightarrow n - \dim Z_\alpha \leq 2d$ ▨

Geometric input :

<u>Prop</u> M : proj$/\bar{k}$ $A^{ab} \curvearrowright M$ w/ finite st. Then

$$\bigoplus_n H^n(M_{\bar{k}}, \overline{\mathbb{Q}}_\ell)[-n]$$

is a free graded Λ_A-module.

Pf) Consider the quotient map

$$M \xrightarrow[\text{sm proj}]{m} N := M/A \xrightarrow[\text{DM}]{\text{prop}} *$$

<u>Claim</u> : $R^i m_* \overline{\mathbb{Q}}_\ell$ is a trivial loc. sys.

Pf)
$$\begin{array}{ccc} A\times M & \xrightarrow{a} & M \\ q\downarrow & \square & \downarrow m \\ M & \xrightarrow{m} & N \end{array}$$

$\Rightarrow m^* R^i m_* \overline{\mathbb{Q}}_\ell \simeq R^i q_* q^* \overline{\mathbb{Q}}_\ell$ as A-equiv. loc sys. (sm base change)

Translation action on $H^*(A, \overline{\mathbb{Q}}_\ell)$ is trivial

$\Rightarrow R^i q_* q^* \overline{\mathbb{Q}}_\ell$: trivial loc. sys. w/ triv A-equiv. □

Leray s.s :

$$H^j(N, R^i m_* \overline{\mathbb{Q}}_\ell) \Rightarrow H^{i+j}(M, \overline{\mathbb{Q}}_\ell)$$

$$gr^j_F \overset{(\text{claim})}{\simeq} H^j(N, \mathbb{Q}_\ell) \otimes (\bigoplus H^i(A, \overline{\mathbb{Q}}_\ell))$$ w/ triv Λ_A-mod

$\rightarrow H^\cdot(M, \overline{\mathbb{Q}}_\ell)$: <u>free graded Λ_A-module</u> ▨

§6. Proof of freeness

Decreasing Induction on $\dim Z_\alpha$

Step 1) $Z_\alpha = S_{\bar{k}}$

$x_\alpha \in S_{\bar{k}}$: general pt $P_{x_\alpha} \curvearrowright M_{x_\alpha}$

$\overset{wt}{\rightsquigarrow} \Lambda_{A_{x_\alpha}} \curvearrowright \bigoplus_i H^i(M_{x_\alpha}, \overline{\mathbb{Q}}_\ell)[-i]$

<u>Lemma</u> k: finite field $0 \to R \to P \to A \to 0$ splits up to Isogeny

$\Rightarrow$ The action is of geometric origin

$\Rightarrow$ Follows from <u>Prop</u>.

<u>Step2</u>: $\forall \alpha'$ with $Z_\alpha \hookrightarrow Z_{\alpha'}$, $\forall m$ $\bigoplus_{n\in\mathbb{Z}} H^m(K^n_{\alpha',x_\alpha})[-n]$:

is free $\Lambda_{A_{x_\alpha}}$-mod.

Induction : K^n_{α',x'_α} : free $\Lambda_{A_{x'_\alpha}}$-module.

Polarizability $\Rightarrow$ $T_{\bar{\mathbb{Q}}_\ell}(A_{x_\alpha}) \xrightarrow{\beta} T_{\bar{\mathbb{Q}}_\ell}(P_{x_\alpha}) \xrightarrow{sp} T_{\bar{\mathbb{Q}}_\ell}(P_{y_{\alpha'}}) \twoheadrightarrow T_{\bar{\mathbb{Q}}_\ell}(A_{y_\alpha})$

Injects & splits. $\rightsquigarrow$ free as $\Lambda_{A_{x_\alpha}}$-module $\square$

<u>Step3</u>: Perverse-Leray ss:

$$H^m({}^pH^n(f_*\bar{\mathbb{Q}}_\ell)_{x_\alpha}) \Rightarrow H^{m+n}(M_{x_\alpha}, \bar{\mathbb{Q}}_\ell)$$

$\rightsquigarrow$ $\Lambda_{A_{x_\alpha}}$-st filt $F^\bullet$ on $\mathbb{H} := \bigoplus_j H^j(M_{x_\alpha})[-j]$ st

$$F^m\mathbb{H}/F^m\mathbb{H} \cong \bigoplus_{\alpha'} \bigoplus_j H^m(K^j_{\alpha',x_\alpha})[-j-m].$$

① $\mathbb{H}$ is free $\Lambda_{A_{x_\alpha}}$-mod (Prop)

② $\forall Z_\alpha \hookrightarrow Z_{\alpha'}$, $\bigoplus_j H^m(K^j_{\alpha',x_\alpha})[-j]$: free $\Lambda_{A_{x_\alpha}}$ (Step2)

③ If $m \neq -\dim V_\alpha$,

skyscraper at x_α

$$\bigoplus_j H^m(K^j_{\alpha,x_\alpha})[-j] = \bigoplus_j H^{m+\dim V_\alpha}(K^j_{\alpha,x_\alpha})[-j] = 0$$

Consider

$$0 \subset F^{n+1}\mathbb{H} \subset F^n\mathbb{H} \subset \mathbb{H}$$

· $\mathbb{H}$: free $\Lambda_{A_{x_\alpha}}$ ($\because$ ①)

F^m/F^{m+1} : free $m \neq n$.

· $F^{n+1}\mathbb{H}$ & $\mathbb{H}/F^n\mathbb{H}$: free $\Lambda_{A_{x_\alpha}}$-mod ($\because$ ② + ③)

Using $\Lambda_{A_{x_\alpha}}$: local ring + ε $\Rightarrow$ $F^n\mathbb{H}/F^{n+1}\mathbb{H}$: free

$$= \underbrace{\left(\bigoplus K^j_{\alpha,x_\alpha}[-j-n]\right)}_{\text{free}} \oplus \left(\bigoplus_{\alpha'\neq\alpha} \bigoplus_j H^n(K^j_{\alpha',x_\alpha})[-j-n]\right)$$

$\Leftarrow$ free (Step 2)