Relations on universal Picard stacks & applications

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* $\mathbb{C}$: base field.

§1. Cycle theory on $\mathfrak{Pic}$.

Let $\mathfrak{M}_{g,n}$ = moduli of prestable (nodal) curves of genus g, n markings

- $\pi : \mathfrak{C}_{g,n} \longrightarrow \mathfrak{M}_{g,n}$: universal curve
- $\mathfrak{Pic}_{g,n}$ = Picard stack of $\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}$. $\mathfrak{Pic}_{g,n} = \coprod_{d\in\mathbb{Z}} \mathfrak{Pic}_{g,n,\underline{d}}$ degree

  ↳ smooth algebraic stack, loc of finite type / $\mathbb{C}$

  $\dim = 4g-4+n$.

__Def__ (Operational Chow) $\mathcal{X}$: alg stack loc finite type / $\mathbb{C}$

$c \in CH^p_{op}(\mathcal{X})$ is class of hom : for finite type scheme B & $\varphi : B \to \mathcal{X}$,

$$c(\varphi) : CH_*(B)_{\mathbb{Q}} \longrightarrow CH_{*-p}(B)_{\mathbb{Q}}$$

+ compatibilities.

We define $R^*(\mathfrak{Pic}) \subseteq CH^*_{op}(\mathfrak{Pic})$:

$$\begin{tikzcd} & \mathcal{L} \arrow[dl] \\ \mathfrak{C}_{g,n} \arrow[d, "\pi"] & \\ \mathfrak{Pic}_{g,n} \arrow[u, bend left, "p_i"] & \end{tikzcd}$$

$\psi_i = p_i^* c_1(\omega_\pi)$, $\xi_i = p_i^* c_1(\mathcal{L})$, $\eta_{ab} = \pi_*(c_1(\omega_{\log})^a c_1(\mathcal{L})^b)$.

$\Gamma_\delta = (\Gamma, \delta)$ $\qquad \mathfrak{Pic}_{\Gamma_\delta} \longrightarrow \mathfrak{Pic}_{g,n}$. $\qquad \boxed{\eta := \eta_{0,2}}$

(Γ: prestable graph, δ: degree)

__Def__. $R^*(\mathfrak{Pic}_{g,n}) = \mathbb{Q}\langle [\Gamma_\delta, \gamma] \rangle \subset CH^*_{op}(\mathfrak{Pic}_{g,n})$.

§2. Pixton's formula.

X : nonsingular projective variety / $\mathbb{C}$, L : line bundle.

$\beta \in H_2(X, \mathbb{Z})$. $d = \int_\beta c_1(L)$. $A = (a_1 \cdots a_n) \in \mathbb{Z}^n$ st $\sum_i a_i = d$.

- X-DR.

$$\{ f : C \to X \mid f^*L \cong \mathcal{O}(\sum a_i p_i) \} \subset \mathcal{M}_{g,n}(X,\beta)$$

$DR_{g,A}(X,L)$: compactification of this locus in $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

For each Γ_δ, a weighting is $w : H(\Gamma_\delta) \to \{0, \cdots, r-1\}$ $r \in \mathbb{N}$ st

$w(i) \equiv a_i \pmod r$ $\quad \underset{e=(h,h')\text{ edge}}{w(h) + w(h') \equiv 0 \pmod r}$, $\sum_{h \mapsto v} w(h) = \delta_v \pmod r$

let

$$P^{c,r}_{g,A,d} = \sum_{\Gamma_\delta} \sum_w \frac{1}{|\mathrm{Aut}\,\Gamma_\delta|} \frac{1}{r^{h^1(\Gamma_\delta)}} \Big[\Gamma_\delta, \prod_{v \in V} \exp(-\tfrac{1}{2}\eta_v) \prod_{i=1}^n \exp(\tfrac{1}{2} a_i^2 \psi_i + a_i \xi_i) \prod_{e=(h,h')} \frac{1 - \exp\left(-\frac{w(h)w(h')}{2}(\psi_h + \psi_{h'})\right)}{\psi_h + \psi_{h'}} \Big]_{\mathrm{codim}=c}$$

This expression is polynomial in r as $r \gg 0$.

$\rightsquigarrow P^c_{g,A,d} := [P^{c,r}_{g,A,d}]_{r^0}$.

For (X,L), $\varphi_L : \overline{\mathcal{M}}_{g,n}(X,\beta) \longrightarrow \mathrm{Pic}_{g,n}$ $[f : C \to X] \mapsto (C, f^*L)$

<u>Thm</u> 1) [JPPZ '19] $P^g_{g,A,d}(\varphi_L) \cap [\overline{\mathcal{M}}(X)]^{vir} = DR_{g,A}(X,L)$.

2) [B'20, Fan-Wu-You] $P^c_{g,A,d}(\varphi_L) \cap [\overline{\mathcal{M}}(X)]^{vir} = 0$ for $c \geq g+1$

Q) $P^{\geq g+1}_{g,A,d} = 0$ in $CH^{\geq g+1}_{op}(\mathrm{Pic}_{g,n})$?

Thm [Morita '89] Restricting to the locus C = smooth,

$$P^{\geq g+1}_{g,\phi,0} = 0 \quad \text{in } H^*(\mathrm{Pic}^{sm}_{g,\phi,0}),\ g \geq 2.$$

Thm [B-Holmes-Pandharipande-Schmitt-Schwartz]

$$P^{\geq g+1}_{g,A,d} = 0 \quad \text{in } CH^*_{op}(\mathrm{Pic}_{g,n}).$$

Remarks 1) $P^c_{g,A,d}\big|_{a_n = d - a_1 - \cdots - a_{n-1}}$ is polynomial in $a_1 \cdots a_{n-1}$.

$\Rightarrow$ get finer relations

2) We can twist relations by the automorphism of Pic.

$$\phi_{k,B} : \mathrm{Pic} \to \mathrm{Pic} \qquad (C, \mathcal{L}) \longmapsto (C, \mathcal{L} \otimes \omega_C^k(\textstyle\sum b_i p_i)).$$

§3. Sketch of the proof.

Idea Reduce to $\overline{M}_{g,n}(\mathbb{P}^\ell, d) \supset \overline{M}_{g,n}(\mathbb{P}^\ell, d)' = \{ f \mid H^1(C, f^*\mathcal{O}(1)) = 0 \}$

On this locus, $[\cdots]^{vir} = [\cdots]$.

For $\varphi : B \to \mathrm{Pic}$, it corresponds to

$$(C \xrightarrow{\pi} B, p_1 \cdots p_n, L)$$

Step 1 When L is sufficiently positive (ie rel. bpf & $R^1\pi_* \mathcal{L} = 0$)

Consider the linear system on B

$$\begin{tikzcd} U_\ell \arrow[r, hook] & E_\ell = (\pi_* \mathcal{L})^{\ell+1} \arrow[d] \\ & B \arrow[ul, phantom] \end{tikzcd}$$

$$U_\ell \xrightarrow{\varphi} B$$

$l \gg 0$, $CH.(B) \xrightarrow{\psi^*} CH.(U_l)$ injective

$$\begin{tikzcd} U_l \arrow[r] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowleft"] & \overline{M}(\mathbb{P}^l)' \arrow[d, "\varphi_L"] \\ B \arrow[r, "\varphi"] & Pic \end{tikzcd}$$

Step 2 When we have enough sections.

Twist $\mathcal{L}$ by $\mathcal{O}(\sum_i a_i p_i)$ $a_i \gg 0$ $\rightsquigarrow$ $P^c_{g,A+A'} = 0$

Now use the automorphism ϕ^*_A to get vanishing of $P^c_{g,A}$ – $(*)$

Step 3. General case.

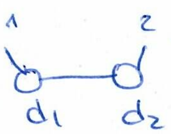
We modify the base: $\exists\ h: B' \to B$ alteration and a family $C' \to B'$

$$\begin{tikzcd} C' \arrow[dr, "f"] \arrow[ddr, bend right, "\pi'"'] \arrow[rr, dashed, bend left, "\mathcal{L}'"] & & \mathcal{L} \arrow[d] \\ & C \times_B B' \arrow[r] \arrow[d] & C \arrow[d, "\pi"] \\ & B' \arrow[r, "h"] & B \end{tikzcd}$$

f: contracting some $\mathbb{P}^1$-chains where $\mathcal{L}$ is trivial.

and $P^c_{g,A}(C', B', L') = h^* P^c_{g,A}(C, B, L)$ — $(**)$ ∎

Example [Lee-Pandharipande] In $g=0$,

$\xi_1 - \xi_2 - d\psi_2 + \sum_{d_1+d_2=d} d_1 [$  $] = 0$

Remarks ① There is a natural morphism.

$$\underbrace{CH_*(Pic)}_{\text{Chow group by Kresch}} \longrightarrow CH^*_{op}(Pic)$$

We could not prove the relation in $CH_*(Pic)$.

② [B-Lho] We have a generalization of 3-spin relations $S_M \in R^*(Pic)$ s.t for any (X, L),

$$S_M(\varphi_L) \cdot [\overline{M}(X)]^{vir} = 0.$$

We could not prove $S_M = 0$ because we are lacking (*), (**) for 3-spin relations.

It is natural to study the structure of $R^*(Pic_{g,n})$.

On $Pic_{g,n}^{sm, rel}$, there are results by Polishchuk, Yin, etc...

The first step would be : what is $R^*(\mathcal{M}_{g,n})$?

Thm [B-Schmitt] $R^*(\mathcal{M}_{0,n}) = CH^*(\mathcal{M}_{0,n})_{\mathbb{Q}}$ and we know all the relations.

↳ Same technique can be generalized to $R^*(Pic_{0,n})$.

For higher genera, it is widely open.

§4. Applications.

We consider GW theory of K3 surfaces.

S: K3 surface. If $\beta \neq 0$, $[\overline{M}(S)]^{vir} = 0$ but

[Bryan-Leung] $[\overline{M}(S)]^{red} \neq 0 \in CH_{g+n}(\overline{M}(S))$.

When β = primitive, much is known by [Maulik-Pandharipande-Thomas]

- $F^S_{g,n}(\cdots) \in \frac{1}{\Delta(q)} \mathrm{QMod}$.

When β = imprimitive, few things are known.

<u>Conjecture</u> [Oberdieck-Pandharipande]

"Imprimitive invariants are determined by primitive invariants".

(by explicit formulas)

evidence:

<u>Thm</u> [Pandharipande-Thomas] <u>Conj</u> holds for λ_g integrals.

(Q) Can we use the structure of $R^*(\mathrm{Pic})$ to upgrade Ionel-Getzler vanishing formula.

In low genus ($g \leq 3$), we checked that

$$\text{MCC for } g \leq 3 \iff \text{MCC for } \langle \tau_0(p)^2 \rangle_{g=2} \text{ and } \langle \tau_0(p)^3 \rangle_{g=3} + \lambda_g\text{-integrals}.$$

When the divisibility is 2, we have the following result:

<u>Thm</u> [B-Bülles] When $\beta = 2$ (primitive),

① $F^{S}_{g,\beta} \in \frac{1}{\Delta(q)^2} \mathrm{QMod}(\Gamma_0(2))$.

② The imprimitive holomorphic anomaly equation holds for $F^{S}_{g,\beta}$

~~be We still don't know~~

- The proof involves imprimitive degeneration formula, λ_g-integrals relations on $R^*(\mathrm{Pic}_{g,n})$ and $R^{g-1}(M_{g,n})$.

- We still don't know the multiple cover conjecture.